AMCS / CS 247 – Scientific Visualization
Lecture 23: Vector Field / Flow Visualization, Pt. 4

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Reading Assignment #12 (until Nov 23)

Read (required):

- Streak Lines as Tangent Curves of a Derived Vector Field,
  Tino Weinkauf and Holger Theisel, IEEE Vis 2010
  http://dx.doi.org/10.1109/TVCG.2010.198

A very brief overview of vector calculus, dynamical systems, and fluid simulation ctd.

Some Vector Calculus (1)

Gradient (scalar field → vector field)

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \right)$$

- Direction of fastest change; magnitude = rate
- *Conservative* vector field: gradient of some scalar function (potential)

Divergence (vector field → scalar field)

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

- Volume density of outward flux:
  "exit rate: source? sink?"
- *Incompressible/solenoidal/divergence-free vector field*: div = 0
  can express as curl (next slide) of some vector function (potential)

Laplacian (scalar field → scalar field)

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}$$

- Divergence of gradient
- Measure for difference between point and its neighborhood

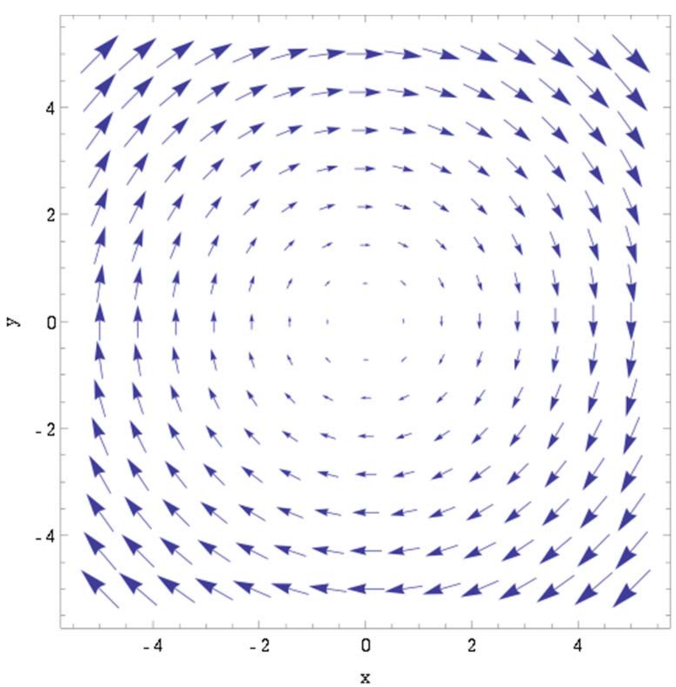
Some Vector Calculus (2)

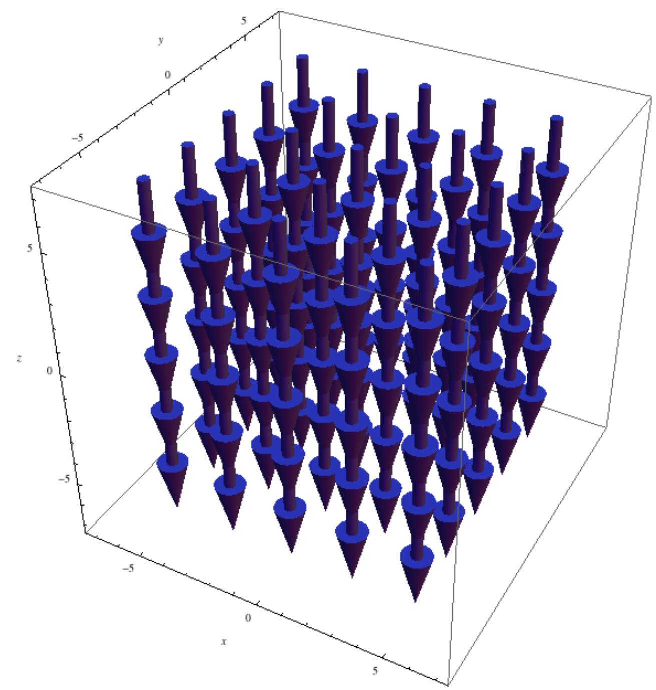
Curl (vector field → vector field)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

- For flow: circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative field is irrotational (conv. true if simply connected)

Example:
curl = const
everywhere





Some Vector Calculus (2)

Curl (vector field → vector field)

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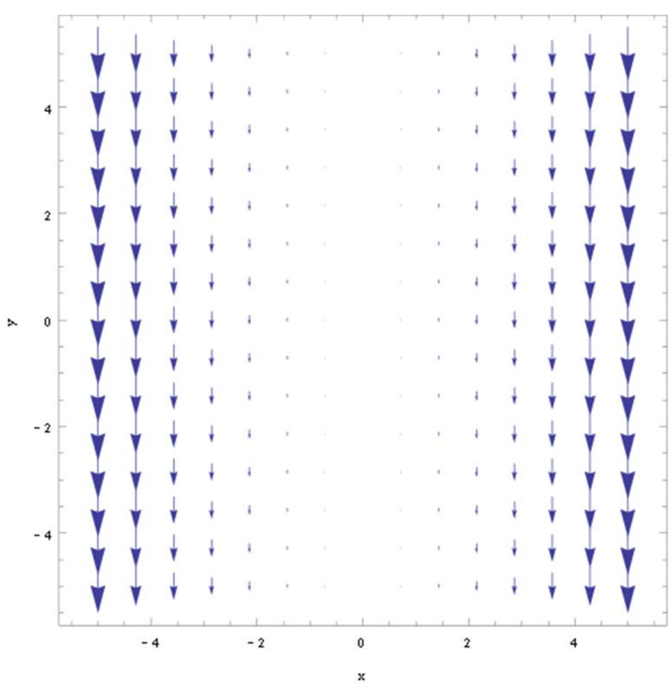
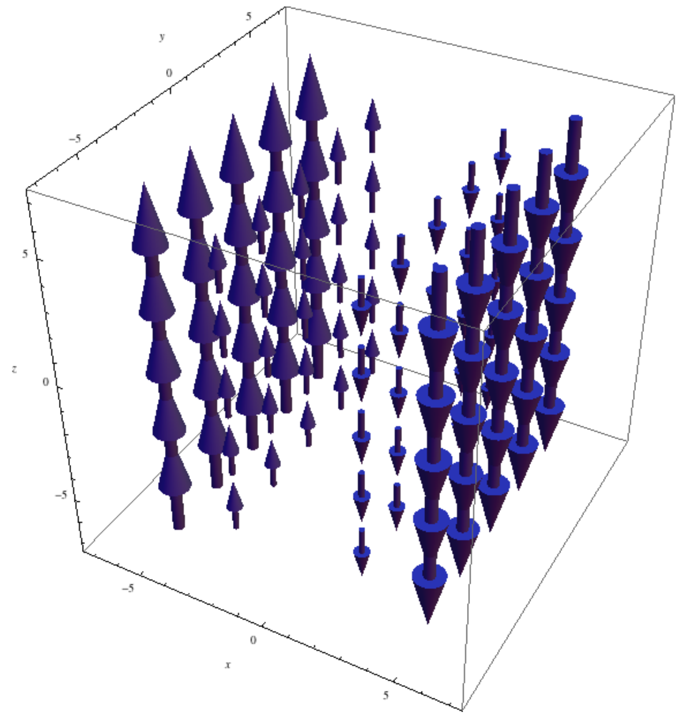
Example: curl not always "obviously rotational"





Some Vector Calculus (2)

Curl (vector field → vector field)

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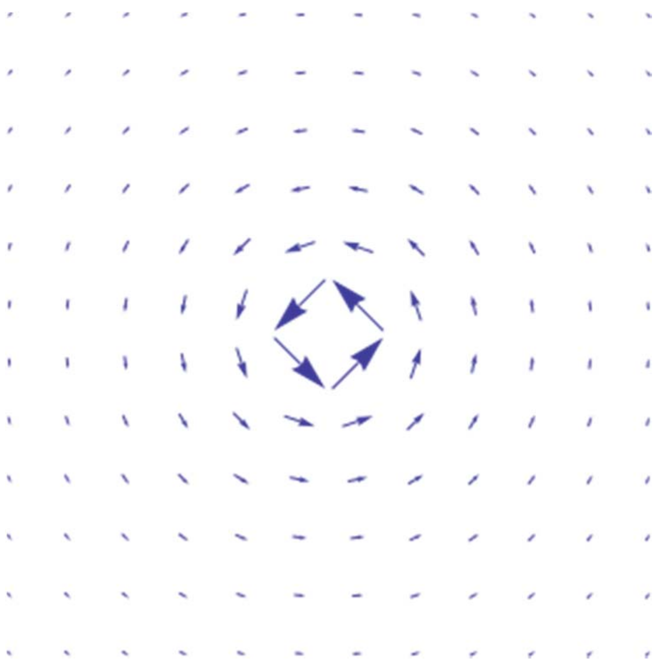
Example:
non-obvious
curl-free field



$$\mathbf{v}(x, y, z) = \frac{(-y, x, 0)}{x^2 + y^2}$$

$$v_x = u_y \qquad \nabla \times \mathbf{v} = 0$$

Jacobian matrix $\mathbf{J}$ is symmetric
(see later)

Some Vector Calculus (2)

Curl (vector field → vector field)

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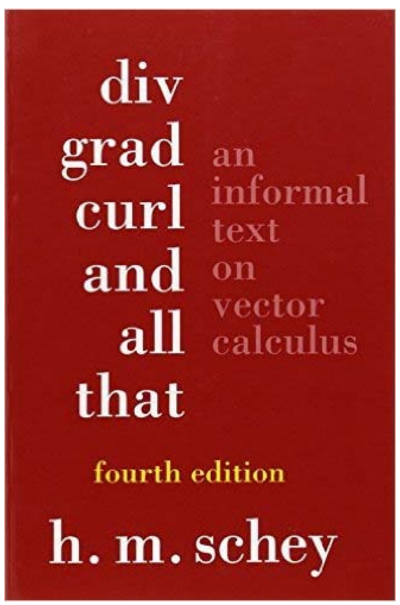
Book:



Interactive tutorial on curl:
`http://mathinsight.org/curl_idea`

Fundamental theorem of vector calculus: any vector field can be expressed as the sum of a solenoidal field and an irrotational field (Helmholtz decomposition)

Fluid Simulation: Navier Stokes (1)

Incompressible (divergence-free) Navier Stokes

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F},$$

$$\nabla \cdot \mathbf{u} = 0,$$

Components:

- Self-advection of velocity (i.e., advection of velocity according to velocity)
- Pressure gradient (force due to pressure differences)
- Diffusion of velocity due to viscosity (for viscous fluids, i.e., not inviscid)
- Application of (arbitrary) external forces, e.g., gravity, user input, etc.

Fluid Simulation: Navier Stokes (2)

Actually, the momentum equation is a system of equations (2 equations in 2D, 3 equations in 3D)

For 2D:

these are PDEs!

$$\frac{\partial u}{\partial t} = -(\mathbf{u} \cdot \nabla) u - \frac{1}{\rho} \nabla p + \nu \nabla^2 u + f_x,$$

$$\frac{\partial v}{\partial t} = -(\mathbf{u} \cdot \nabla) v - \frac{1}{\rho} \nabla p + \nu \nabla^2 v + f_y.$$

Advection

Advection operator, with velocity field **u(**t;*x*,*y*,*z***)**

$$\mathbf{u} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

- Advect a scalar quantity, here: **a(**t;*x*,*y*,*z***)**

$$\frac{\partial \mathbf{a}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{a} = 0.$$

Self-advection of velocity

- Advect scalar components of velocity field individually (actually two equations in 2D, three equations in 3D)

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u}$$

Vector Fields and Dynamical Systems (1)

Jacobian (**matrix**), (vector field → tensor field)

- Gradient of vector field: how fast do the vectors change?
- In our context: also called *velocity gradient tensor*

$$\mathbf{J} = \nabla \mathbf{v} \qquad \mathbf{J}(x,y,z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

- Can be decomposed into *symmetric* part + *antisymmetric* part

$\mathbf{J} = \mathbf{D} + \mathbf{S}$ *velocity gradient tensor*

sym.: $\mathbf{D} = \frac{1}{2} (\mathbf{J} + \mathbf{J}^T)$ strain: *rate-of-strain tensor*

skew-sym.: $\mathbf{S} = \frac{1}{2} (\mathbf{J} - \mathbf{J}^T)$ rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)

Vorticity/spin/angular velocity tensor

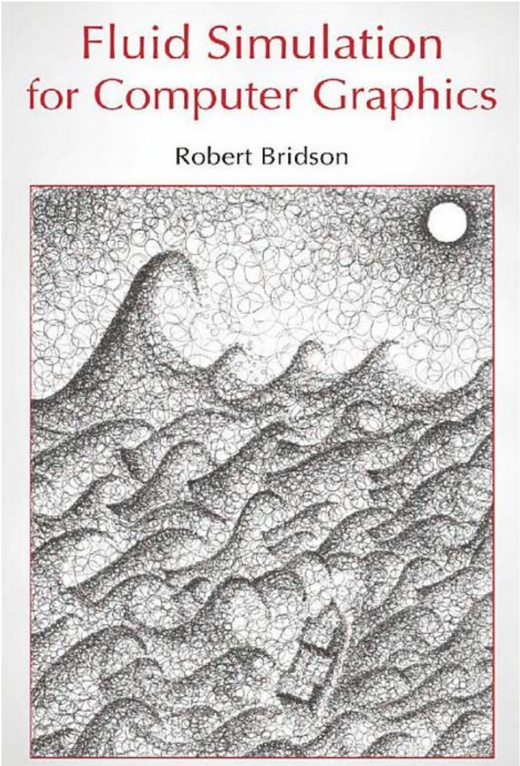
- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor ½)

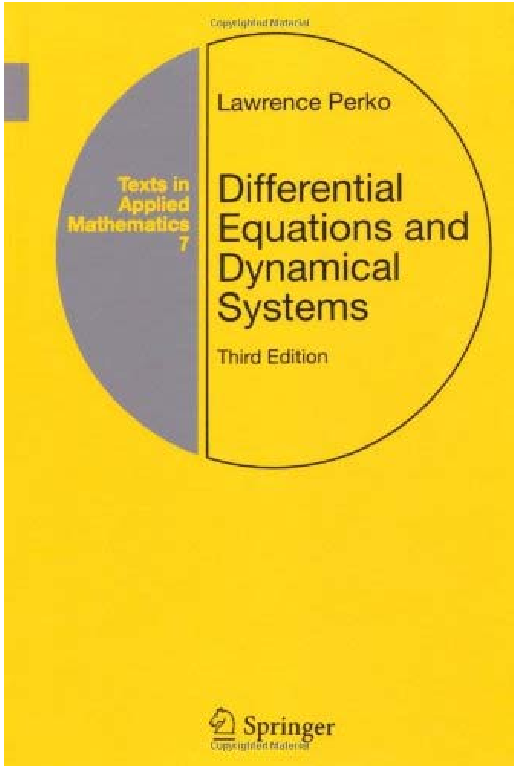
$$\mathbf{S} = \tfrac{1}{2} (\mathbf{J} - \mathbf{J}^T)$$

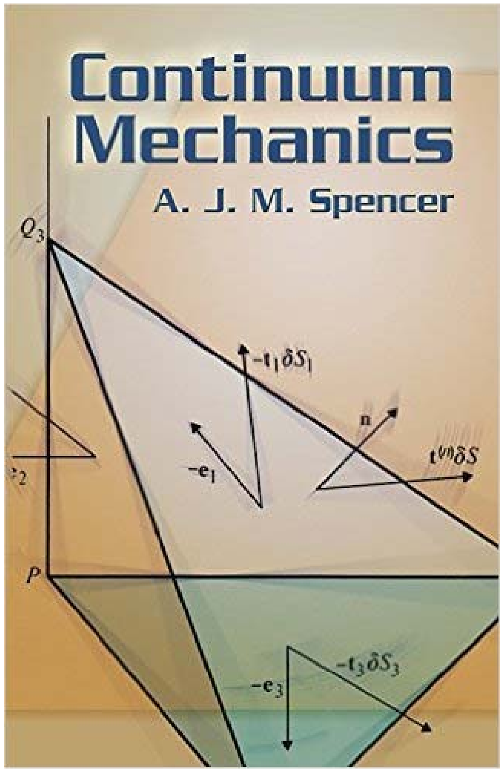
$$\mathbf{S} = \tfrac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

$\mathbf{S}$ acts on vector like cross product with ω: $\mathbf{S} \bullet = \tfrac{1}{2} \omega \times$

Recommended Books







Thank you.

Thanks for material

- Tino Weinkauf
- Holger Theisel
- Ronny Peikert
- Jens Krüger