

# **AMCS / CS 247 – Scientific Visualization**

## **Lecture 24: Vector Field / Flow Visualization, Pt. 5**

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# Reading Assignment #12 (until Nov 23)



Read (required):

- Streak Lines as Tangent Curves of a Derived Vector Field,  
Tino Weinkauff and Holger Theisel, IEEE Vis 2010  
<http://dx.doi.org/10.1109/TVCG.2010.198>

# Quiz #3: Nov 30



## Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

## Content of questions

- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

# Vector Fields and Dynamical Systems (1)



## Jacobian (matrix), (vector field $\rightarrow$ tensor field)

- Gradient of vector field: how fast do the vectors change?
- In our context: also called *velocity gradient tensor*

$$\mathbf{J} = \nabla \mathbf{v} \quad \mathbf{J}(x, y, z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

- Can be decomposed into *symmetric* part + *antisymmetric* part

$$\mathbf{J} = \mathbf{D} + \mathbf{S} \quad \text{velocity gradient tensor}$$

$$\text{sym.:} \quad \mathbf{D} = \frac{1}{2} (\mathbf{J} + \mathbf{J}^T)$$

strain: *rate-of-strain tensor*

$$\text{skew-sym.:} \quad \mathbf{S} = \frac{1}{2} (\mathbf{J} - \mathbf{J}^T)$$

rotation: *vorticity/spin tensor*

# Vector Fields and Dynamical Systems (2)



## Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor  $1/2$ )

$$\mathbf{S} = \frac{1}{2} (\mathbf{J} - \mathbf{J}^T)$$

$$\mathbf{S} = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

$\mathbf{S}$  acts on vector like cross product with  $\boldsymbol{\omega}$ :  $\mathbf{S} \cdot \mathbf{v} = \frac{1}{2} \boldsymbol{\omega} \times \mathbf{v}$

# Vector Fields and Dynamical Systems (3)



## Critical point

- Velocity vanishes (all components zero)

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \epsilon) \neq \mathbf{0}$$

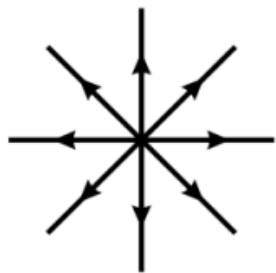
$$\mathbf{v} = \mathbf{0}$$

## Characterize using the Jacobian $\mathbf{J}$ at a point

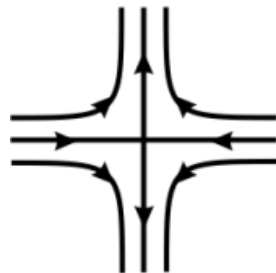
- Look at eigenvalues (and eigenvectors) of  $\mathbf{J}$

$$\mathbf{v} = \mathbf{0}$$

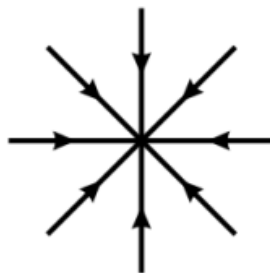
$$\mathbf{J} \quad \det(\mathbf{J}(\mathbf{x}_0)) \neq 0$$



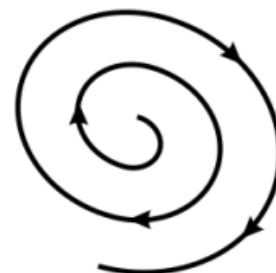
Repelling node  
 $R_1, R_2 > 0$   
 $I_1 = I_2 = 0$



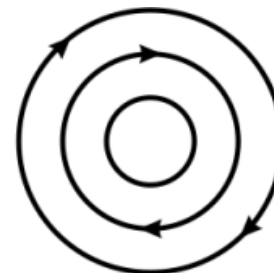
Saddle point  
 $R_1 < 0, R_2 > 0$   
 $I_1 = I_2 = 0$



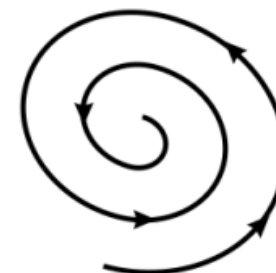
Attracting node  
 $R_1, R_2 < 0$   
 $I_1 = I_2 = 0$



Repelling focus  
 $R_1 = R_2 > 0$   
 $I_1 = -I_2 \neq 0$



Center  
 $R_1 = R_2 = 0$   
 $I_1 = -I_2 \neq 0$



Attracting focus  
 $R_1 = R_2 < 0$   
 $I_1 = -I_2 \neq 0$

# (Non-Linear) Dynamical Systems



Start with system of linear ODEs

- Non-linear systems can be linearized around a point
- Use linearization for characterization

$$\dot{\mathbf{x}} = A\mathbf{x} \quad A \text{ is an } n \times n \text{ matrix}$$

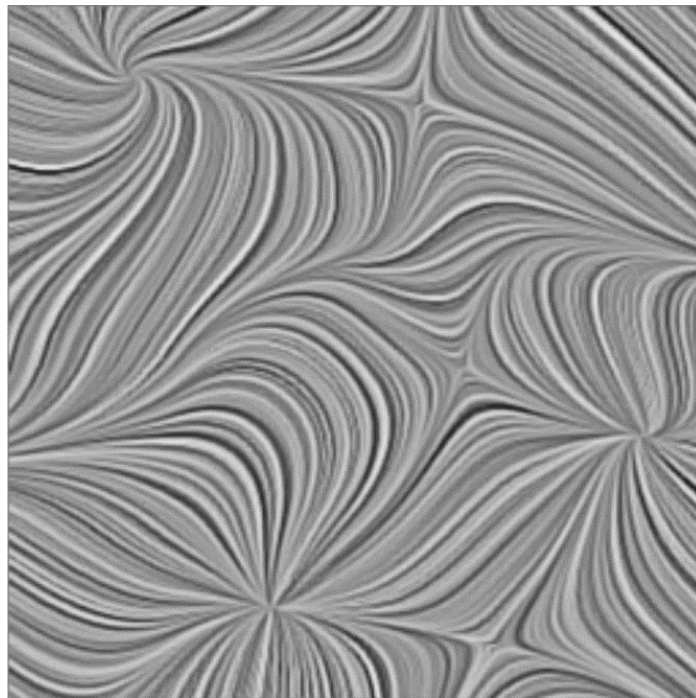
$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

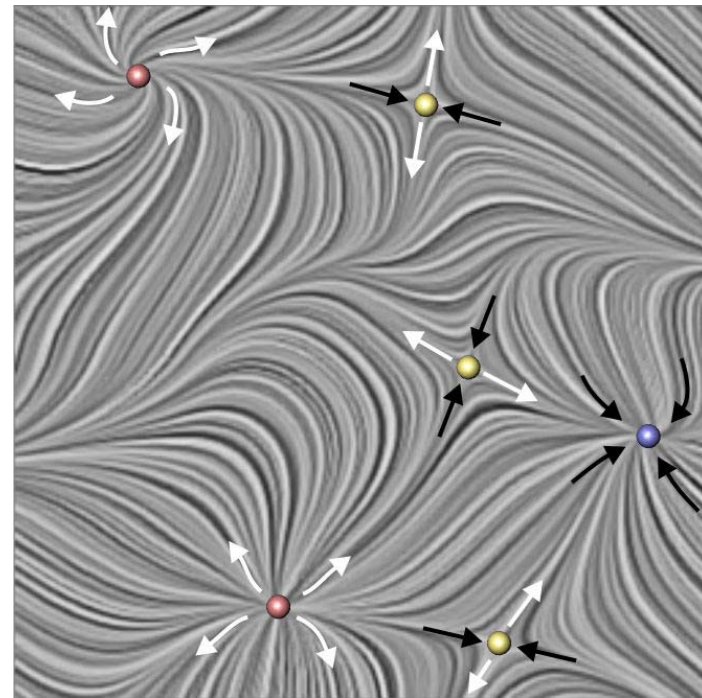
$$\text{solution: } \mathbf{x}(t) = e^{At}\mathbf{x}_0$$

characterize behavior  
through eigenvalues of A

# Critical Points



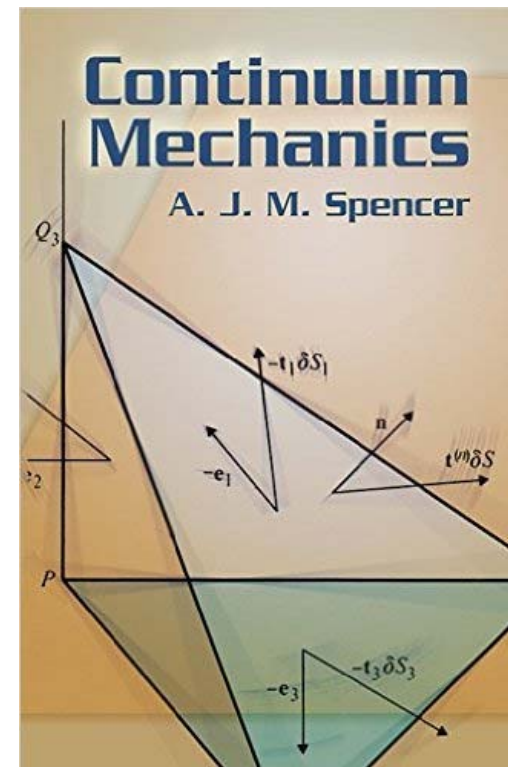
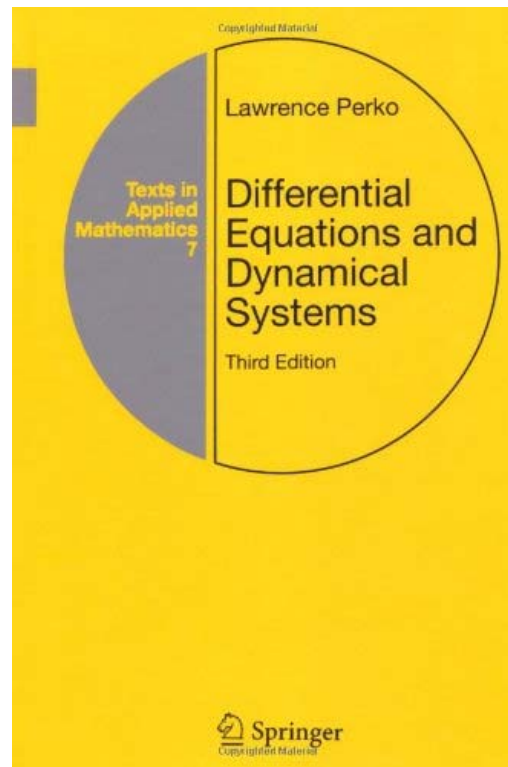
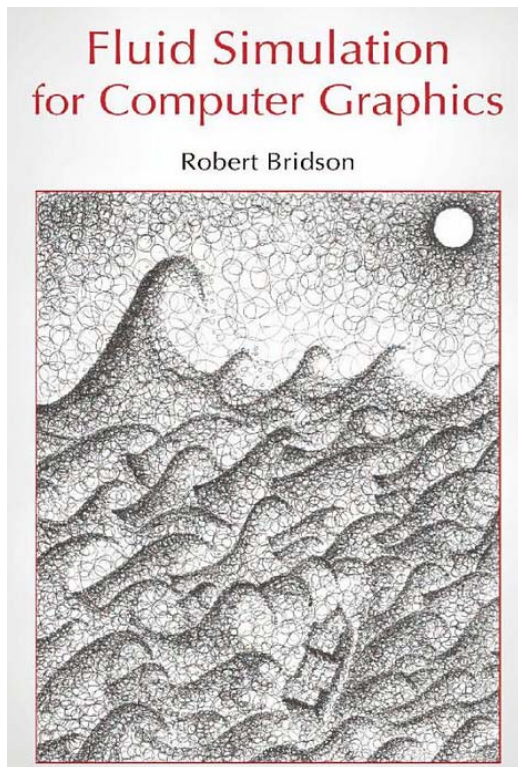
stream lines



critical points



# Recommended Books



# Thank you.

Thanks for material

- Tino Weinkauf
- Holger Theisel
- Ronny Peikert
- Jens Krüger