AMCS / CS 247 – Scientific Visualization
Lecture 27: Vector Field / Flow Visualization, Pt. 8

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Reading Assignment #14 (until Dec 7)

Read (required):

- B. Jobard, G. Erlebacher, M. Y. Hussaini: *Lagrangian-Eulerian Advection of Noise and Dye Textures for Unsteady Flow Visualization*, TVCG 8(3), 2002

  `http://dx.doi.org/10.1109/TVCG.2002.1021575`

- J. van Wijk: *Image-Based Flow Visualization*, ACM SIGGRAPH 2002

  `http://www.win.tue.nl/~vanwijk/ibfv/ibfv.pdf`

Vector Fields and Dynamical Systems (1)

Jacobian (**matrix**), (vector field → tensor field)

- Gradient of vector field: how fast do the vectors change?
- In our context: also called *velocity gradient tensor*

$$\mathbf{J} = \nabla \mathbf{v} \qquad \mathbf{J}(x,y,z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

- Can be decomposed into *symmetric* part + *antisymmetric* part

$\mathbf{J} = \mathbf{D} + \mathbf{S}$ *velocity gradient tensor*

sym.: $\mathbf{D} = \frac{1}{2} (\mathbf{J} + \mathbf{J}^T)$ strain: *rate-of-strain tensor*

skew-sym.: $\mathbf{S} = \frac{1}{2} (\mathbf{J} - \mathbf{J}^T)$ rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)

Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor ½)

$$\mathbf{S} = \tfrac{1}{2} (\mathbf{J} - \mathbf{J}^{\mathrm{T}})$$

$$\mathbf{S} = \tfrac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

$\mathbf{S}$ acts on vector like cross product with ω : $\mathbf{S} \bullet = \tfrac{1}{2} \omega \times$

$$\mathbf{v}^{(r)} = \mathbf{S} \cdot d\mathbf{r} = \frac{1}{2} \left[\nabla \mathbf{v} - (\nabla \mathbf{v})^T \right] \cdot d\mathbf{r} = \frac{1}{2} \boldsymbol{\omega} \times d\mathbf{r}$$

A Few Facts about Eigenvalues and –vectors

The matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ has eigenvalues $\lambda_1 = c + s\mathbf{i} \quad \lambda_2 = c - s\mathbf{i}$

with eigenvectors $u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$

If c = 0, this is a skew-symmetric matrix

Skew-symmetric matrices are “infinitesimal rotations”

If $c = \cos\theta$ and $s = \sin\theta$ this is a 2x2 rotation matrix:

$$\begin{aligned} \lambda_1 = e^{\mathbf{i}\theta} &= \cos\theta + \mathbf{i}\sin\theta \\ \lambda_2 = e^{-\mathbf{i}\theta} &= \cos\theta - \mathbf{i}\sin\theta \end{aligned}$$

Eigenvalues

- Symmetric matrix: all eigenvalues are *real*
- Skew-symmetrix matrix: all eigenvalues are *pure imaginary*

Vector Fields and Dynamical Systems (3)

Linear: System of linear ODEs

- Non-linear systems can be linearized around a point
- Use linearization for characterization

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A \text{ is an } n \times n \text{ matrix}$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

solution: $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

characterize behavior
through eigenvalues of A

Vector Fields and Dynamical Systems (4)

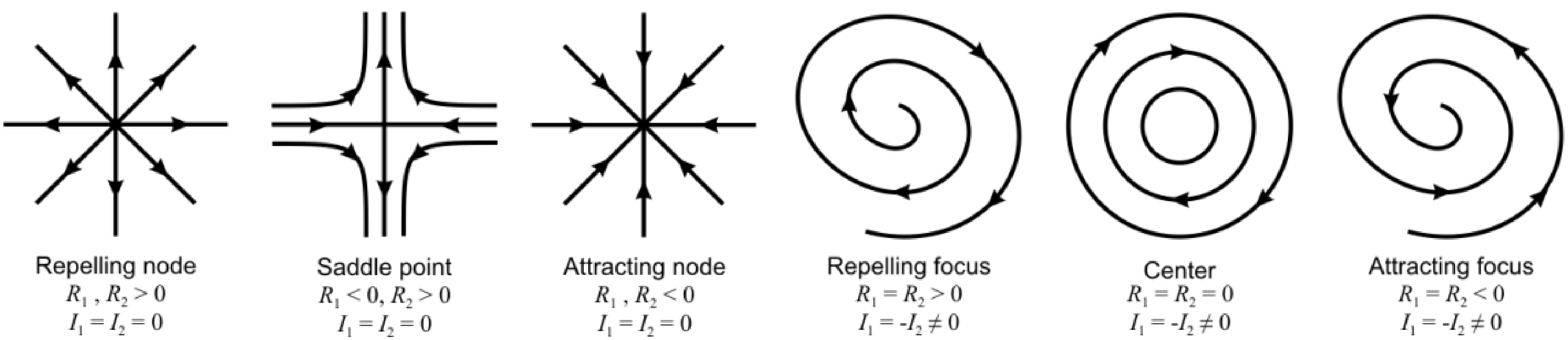
Critical point

- Velocity vanishes (all components zero) $\mathbf{v} = \mathbf{0}$

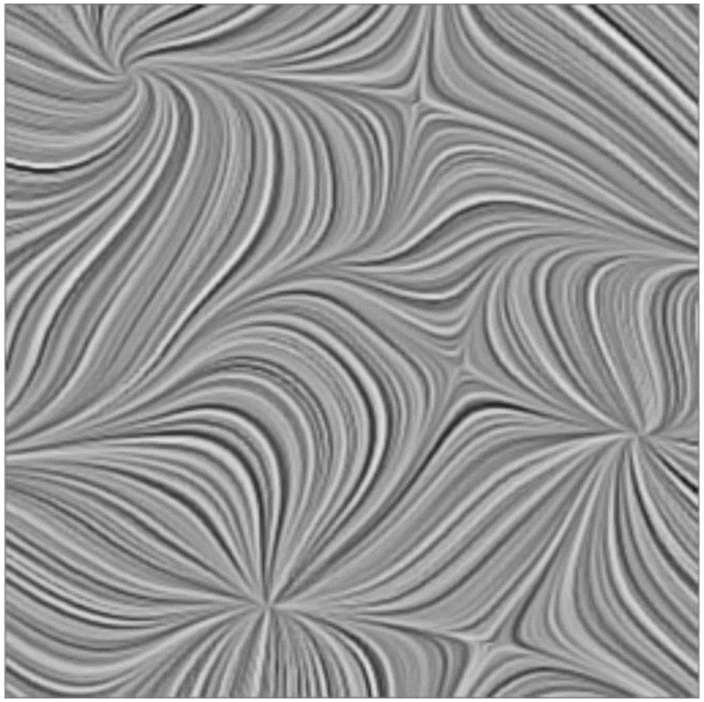
$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$

Characterize using the Jacobian $\mathbf{J}$ at a point $\mathbf{v} = \mathbf{0}$

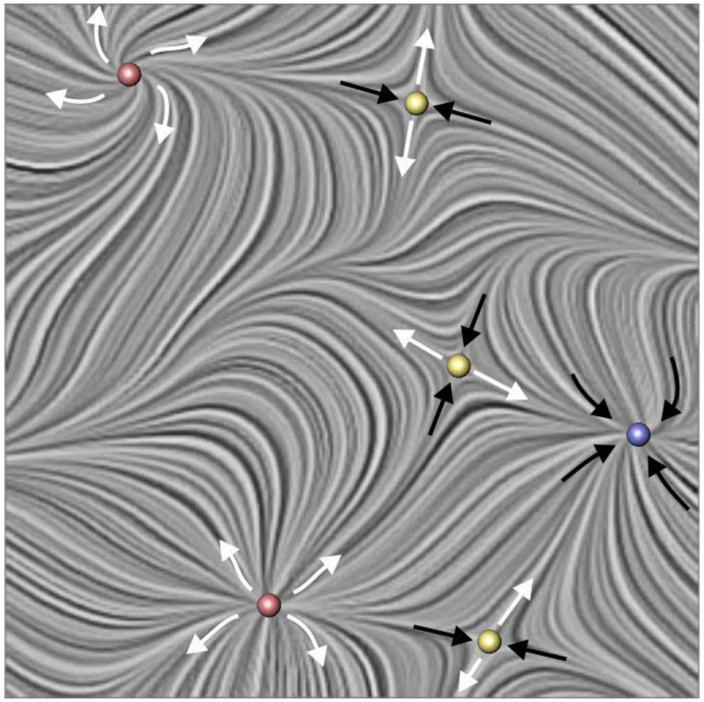
- Look at eigenvalues (and eigenvectors) of $\mathbf{J}$ $\quad \mathbf{J} \quad \det(\mathbf{J}(\mathbf{x}_0)) \neq 0$



Critical Points

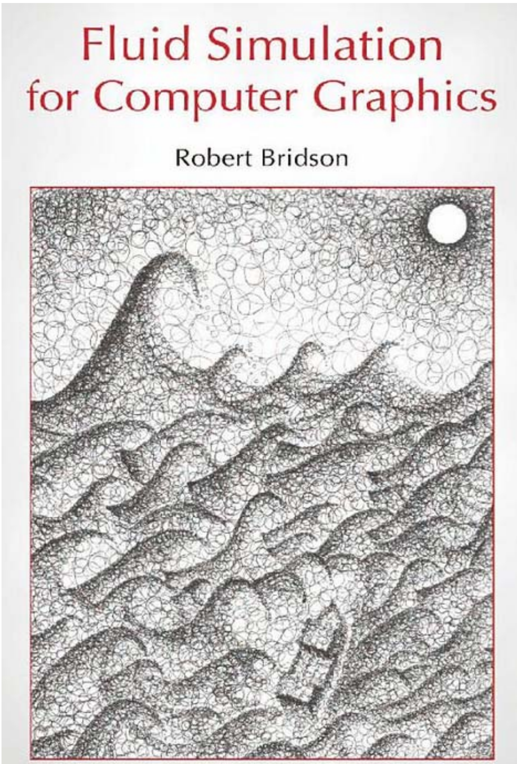


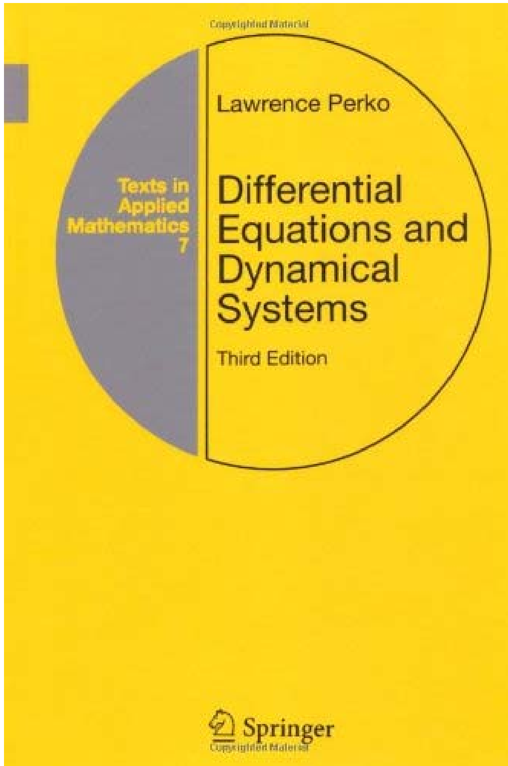
stream lines

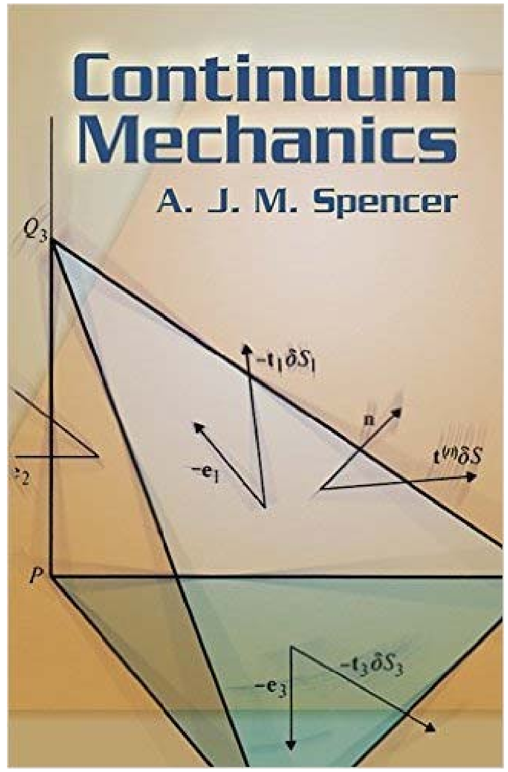


critical points

Recommended Books







Thank you.

Thanks for material

- Tino Weinkauf
- Holger Theisel
- Ronny Peikert