

AMCS / CS 247 – Scientific Visualization

Lecture 27: Vector Field / Flow Visualization, Pt. 8

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Reading Assignment #14 (until Dec 7)



Read (required):

- B. Jobard, G. Erlebacher, M. Y. Hussaini: *Lagrangian-Eulerian Advection of Noise and Dye Textures for Unsteady Flow Visualization*, TVCG 8(3), 2002

<http://dx.doi.org/10.1109/TVCG.2002.1021575>

- J. van Wijk: *Image-Based Flow Visualization*, ACM SIGGRAPH 2002

<http://www.win.tue.nl/~vanwijk/ibfv/ibfv.pdf>

Vector Fields and Dynamical Systems (1)



Jacobian (matrix), (vector field \rightarrow tensor field)

- Gradient of vector field: how fast do the vectors change?
- In our context: also called *velocity gradient tensor*

$$\mathbf{J} = \nabla \mathbf{v} \quad \mathbf{J}(x, y, z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

- Can be decomposed into *symmetric* part + *antisymmetric* part

$$\mathbf{J} = \mathbf{D} + \mathbf{S} \quad \text{velocity gradient tensor}$$

$$\text{sym.:} \quad \mathbf{D} = \frac{1}{2} (\mathbf{J} + \mathbf{J}^T)$$

strain: *rate-of-strain tensor*

$$\text{skew-sym.:} \quad \mathbf{S} = \frac{1}{2} (\mathbf{J} - \mathbf{J}^T)$$

rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)



Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor $1/2$)

$$\mathbf{S} = \frac{1}{2} (\mathbf{J} - \mathbf{J}^T)$$

$$\mathbf{S} = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

\mathbf{S} acts on vector like cross product with $\boldsymbol{\omega}$: $\mathbf{S} \cdot \mathbf{v} = \frac{1}{2} \boldsymbol{\omega} \times \mathbf{v}$

$$\mathbf{v}^{(r)} = \mathbf{S} \cdot d\mathbf{r} = \frac{1}{2} [\nabla \mathbf{v} - (\nabla \mathbf{v})^T] \cdot d\mathbf{r} = \frac{1}{2} \boldsymbol{\omega} \times d\mathbf{r}$$

A Few Facts about Eigenvalues and –vectors



The matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ has eigenvalues $\lambda_1 = c + s\mathbf{i}$ $\lambda_2 = c - s\mathbf{i}$
with eigenvectors $u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$

If $c = 0$, this is a skew-symmetric matrix

Skew-symmetric matrices are “infinitesimal rotations”

If $c = \cos \theta$ and $s = \sin \theta$ this is a 2x2 rotation matrix:

$$\lambda_1 = e^{i\theta} = \cos \theta + \mathbf{i} \sin \theta$$
$$\lambda_2 = e^{-i\theta} = \cos \theta - \mathbf{i} \sin \theta$$

Eigenvalues

- Symmetric matrix: all eigenvalues are *real*
- Skew-symmetric matrix: all eigenvalues are *pure imaginary*

Vector Fields and Dynamical Systems (3)



Linear: System of linear ODEs

- Non-linear systems can be linearized around a point
- Use linearization for characterization

$$\dot{\mathbf{x}} = A\mathbf{x} \quad A \text{ is an } n \times n \text{ matrix}$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

$$\text{solution: } \mathbf{x}(t) = e^{At}\mathbf{x}_0$$

characterize behavior
through eigenvalues of A

Vector Fields and Dynamical Systems (4)



Critical point

- Velocity vanishes (all components zero)

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \epsilon) \neq \mathbf{0}$$

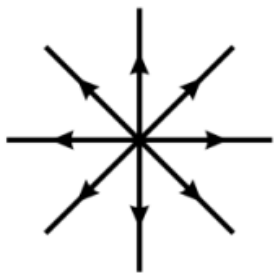
$$\mathbf{v} = \mathbf{0}$$

Characterize using the Jacobian \mathbf{J} at a point

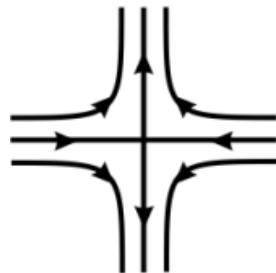
- Look at eigenvalues (and eigenvectors) of \mathbf{J}

$$\mathbf{v} = \mathbf{0}$$

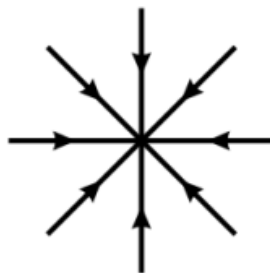
$$\mathbf{J} \quad \det(\mathbf{J}(\mathbf{x}_0)) \neq 0$$



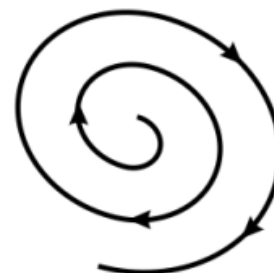
Repelling node
 $R_1, R_2 > 0$
 $I_1 = I_2 = 0$



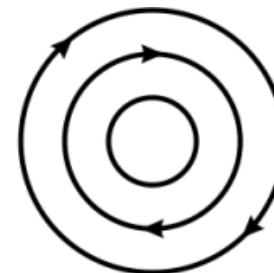
Saddle point
 $R_1 < 0, R_2 > 0$
 $I_1 = I_2 = 0$



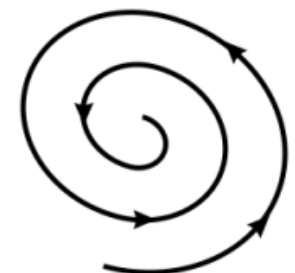
Attracting node
 $R_1, R_2 < 0$
 $I_1 = I_2 = 0$



Repelling focus
 $R_1 = R_2 > 0$
 $I_1 = -I_2 \neq 0$

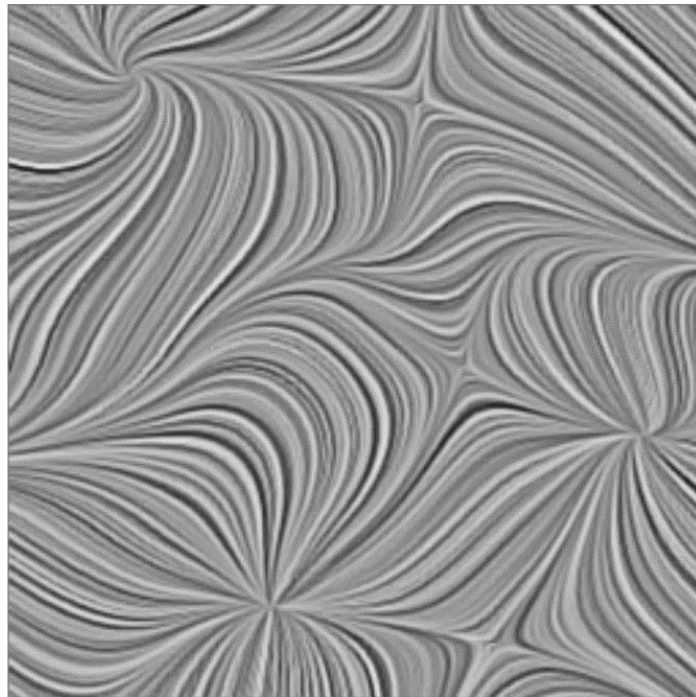


Center
 $R_1 = R_2 = 0$
 $I_1 = -I_2 \neq 0$

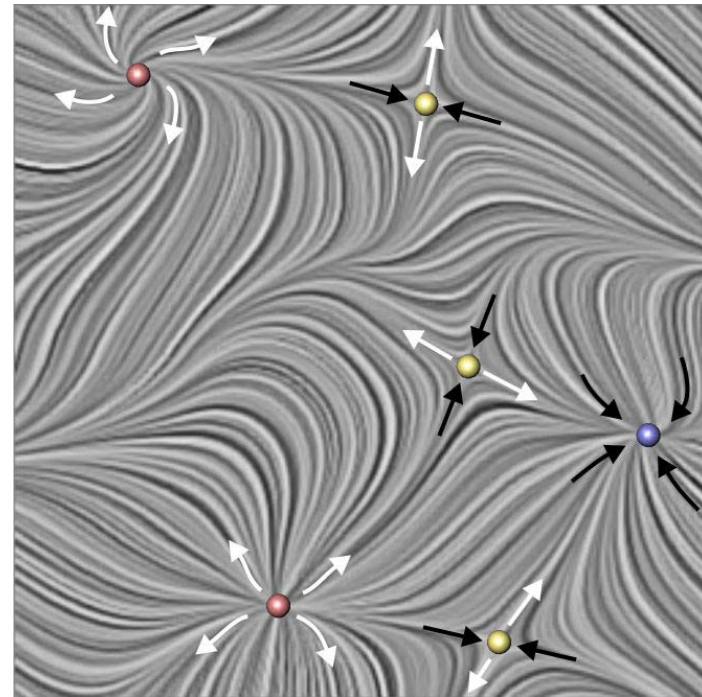


Attracting focus
 $R_1 = R_2 < 0$
 $I_1 = -I_2 \neq 0$

Critical Points

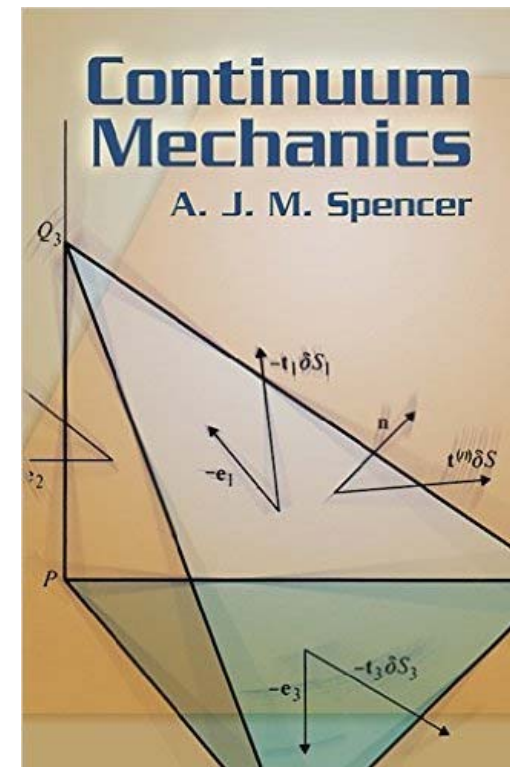
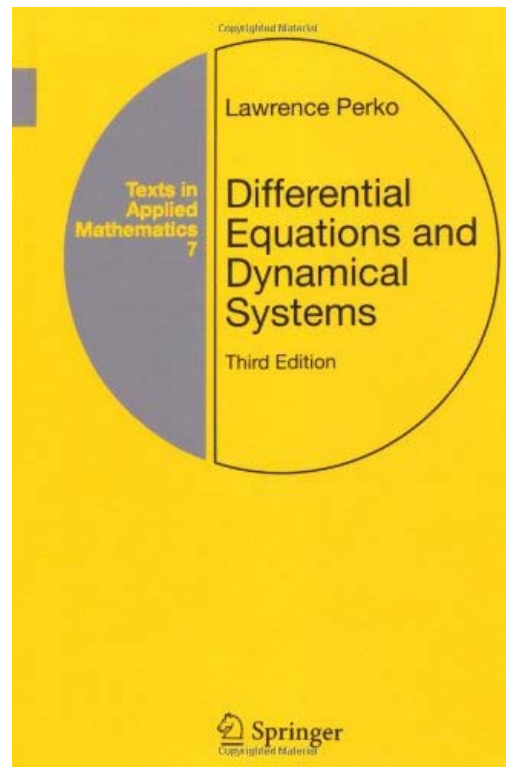
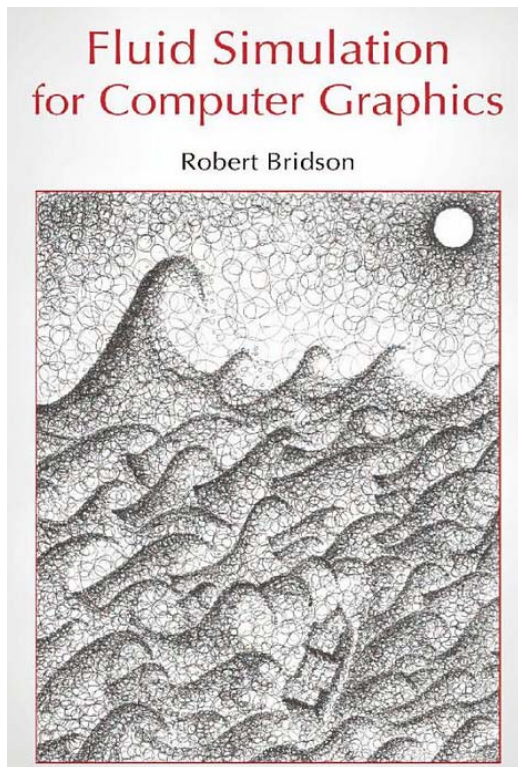


stream lines



critical points

Recommended Books



Thank you.

Thanks for material

- Tino Weinkauf
- Holger Theisel
- Ronny Peikert